Girls Get Curves

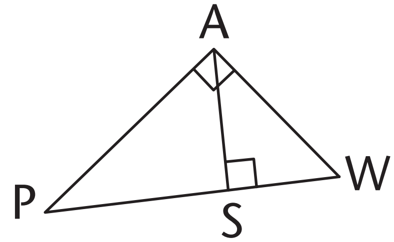
A Proof of the Pythagorean Theorem

(As promised on p. 298 in *Girls Get Curves*)

We first saw the Pythagorean Theorem on p. 176 of *Girls Get Curves*. It says that for a right triangle with legs of lengths a and b, and hypotenuse of length c, then this is true:

$$a^2 + b^2 = c^2$$

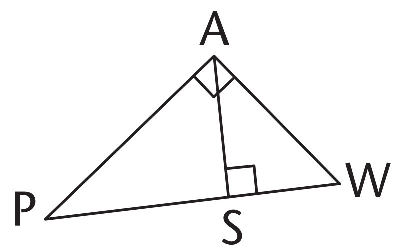
There are many proofs of the Pythagorean theorem; here we'll do one of them! In chapter 17 of *Girls Get Curves*, you might recall that we saw this lovely little diagram, PAWS, consisting of the right triangle $\triangle PAW$ and its altitude, $\overline{AS}$, drawn in.



We'll be proving the Pythagorean theorem using the triangle $\triangle PAW$; in other words, we'll be proving that $(PA)^2 + (AW)^2 = (PW)^2$. That will be our end goal – just good to keep in mind.

Here's our overall strategy: We'll prove two sets of similar triangles, and then use their proportional corresponding sides to get some products! Those products will somehow lead us to the equation we desire: $\boldsymbol{(PA)^2 + (AW)^2 = (PW)^2}$.
This proof is multi-step, so just read along and hang in there!

So, on p. 294, we proved that $\triangle PAW \sim \triangle ASW$. Here's how we did it:

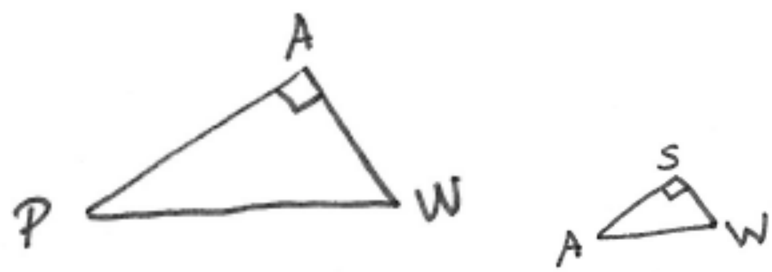


Given: $\triangle PAW$ *is a right triangle, with right angle* $\angle PAW$ *and altitude* $\overline{AS}$. *Prove:* $\triangle PAW \sim \triangle ASW$.

<u>**Statements**</u>	<u>**Reasons**</u>
1. $\triangle PAW$ is a right triangle, with right angle $\angle PAW$.	1. Given
2. $\overline{AS}$ is an altitude	2. Given
3. $\angle ASW$ is a right angle	3. Altitudes create right $\angle$'s
4. $\angle PAW \cong \angle ASW$	4. All right $\angle$'s are $\cong$. (Gimmie an "A"!)
5. $\angle W \cong \angle W$	5. Reflexive Property (Gimmie another "A"!)
6. $\therefore$ $\triangle PAW \sim \triangle ASW$	6. AA

Now, because $\triangle PAW \sim \triangle ASW$, that means its corresponding sides have equal ratios. (See p. 286 to review this!) For example, this means that $\dfrac{PW}{AW} = \dfrac{AW}{SW}$.

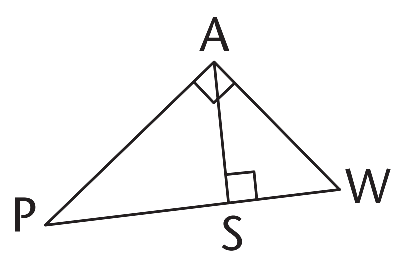
Let's draw the big and small triangles side-by-side and in the same orientation, so the correspondence is easier to see. We'll "lift" the little triangle and then flip and rotate it in order to do this. (See chapter 7 for more on transformations!)



Now I bet it's easier to see why $\frac{PW}{AW} = \frac{AW}{SW}$!

Taking the cross-product, we get the true statement: $\boldsymbol{(AW)^2 = PW \cdot SW}$**.**

We'll come back to that bolded equation in a moment. For now, let's do the same exact thing for the big and medium triangles – which we also started in the book. In fact, in #2 of the exercises on p. 298, we proved that $\triangle PAW \sim \triangle PSA$. Here's that proof:



Given: $\triangle PAW$ *is a right triangle, with right angle* $\angle PAW$ *and altitude* $\overline{AS}$.

Prove $\triangle PSA \sim \triangle PAW$.

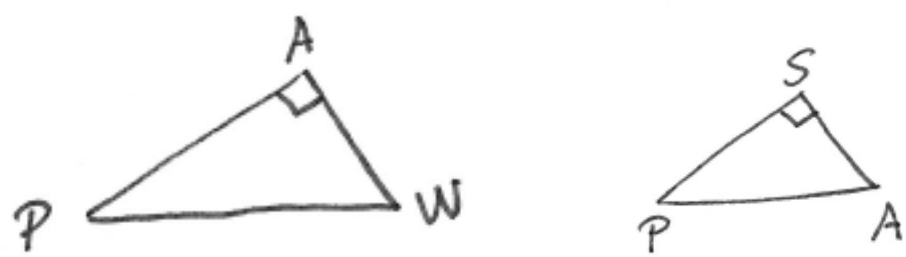
♥Proof♥

Statements	**Reasons**
1. $\triangle PAW$ is a right triangle, with right angle $\angle PAW$.	1. Given
2. $\overline{AS}$ is an altitude	2. Given

3. $\angle PSA$ is a right angle	3. Definition of altitude (altitudes create right angles)
4. $\angle PAW \cong \angle PSA$	4. All right angles are congruent (Gimmie an "A"!)
5. $\angle P \cong \angle P$	5. Reflexive Property (Gimmie another "A"!)
6. $\therefore \triangle PAW \sim \triangle PSA$	6. AA

So at this point, we've proven that $\triangle PAW \sim \triangle PSA$. Great![1]

Again, similar triangles means that their corresponding sides have equal ratios. For example, this means that $\frac{PW}{PA} = \frac{PA}{PS}$. Do you see how those are corresponding sides? Let's draw them separately; we'll take the medium triangle, $\triangle PSA$, and flip & rotate it in order to orient it in the same direction as the big triangle, $\triangle PAW$. It just makes the correspondence much easier to see!



Now do you see how $\frac{PW}{PA} = \frac{PA}{PS}$? Once you do, then keep reading!

[1] Recall that on p. 294 of *Girls Get Curves*, we proved that $\triangle PAW \sim \triangle ASW$, so by the Transitive Property, we know that $\triangle ASW \sim \triangle PSA$, which means all three right triangles are similar to each other! This factoid isn't needed for this proof, but it's just good to know that's what happens when we draw an altitude to the hypotenuse of a right triangle.

Taking the cross-product of $\frac{PW}{PA} = \frac{PA}{PS}$, we get this true statement: $\boldsymbol{(PA)^2 = PW \cdot PS}$.

So far so good? Great!

Remember from 2 pages ago we found this other true statement: $\boldsymbol{(AW)^2 = PW \cdot SW}$.

Our next move will be to take the two bolded statements above and somehow combine them it in a way that gives us the Pythagorean Theorem. It's a little sneaky, and here's how it goes. Remember from algebra that we're allowed to add two equations and get a third true statement? For example, if we know that $a = 7$ and $b = 8$, we can add these to get another true statement: $a + b = 7 + 8$, or:

$$
\begin{array}{ll}
 & a = 7 \\
 & b = 8 \\
+ & \text{-------} \\
 & a + b = 7 + 8
\end{array}
$$

In the same way, we can add our two bolded statements and we'll get a third, true statement:

$$
\begin{array}{ll}
 & \boldsymbol{(PA)^2 = PW \cdot PS} \\
 & \boldsymbol{(AW)^2 = PW \cdot SW} \\
+ & \text{-------------------------} \\
 & \boldsymbol{(PA)^2 + (AW)^2 = PW \cdot PS + PW \cdot SW}
\end{array}
$$

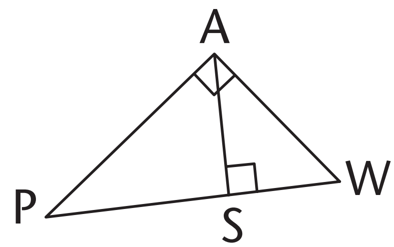
Now let's factor the right side of this equation – after all, it's got a common factor of PW! *(See Chapter 3 in Hot X: Algebra Exposed to review factoring variables out of expressions – and pulling out of creepy parties.)*

So we get:

$$\boldsymbol{(PA)^2 + (AW)^2 = PW \cdot PS + PW \cdot SW}$$

(factoring out PW from the right side) →

$$\boldsymbol{(PA)^2 + (AW)^2 = PW(PS + SW)}$$



But looking at our diagram again, notice that $(PS + SW) = PW$, so we can rewrite the equation again as $\boldsymbol{(PA)^2 + (AW)^2 = PW(PW)}$; in other words:

$$\boldsymbol{(PA)^2 + (AW)^2 = (PW)^2}$$

And lookie there! With the help of similar triangles and a little sneaky algebra, we've just PROVEN that for the triangle $\triangle PAW$: (longer leg)^2 + (shorter leg)^2 = (hypotenuse)^2

And that, my friends, is the Pythagorean Theorem!