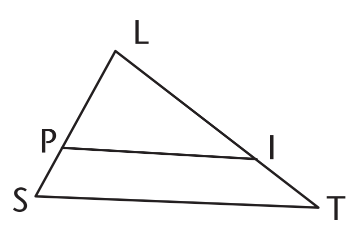
Girls Get Curves

A Proof of the Side-Splitter Theorem

(As promised in the footnote of p. 298 in *Girls Get Curves*)

The Side-Splitter Theorem: If a line is parallel to one side of a triangle and intersects the other two sides, then it divides those two sides proportionally.

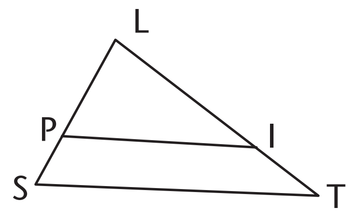
For example, in the SPLIT diagram below, if $\overline{PI} \parallel \overline{ST}$, then $\frac{PL}{PS} = \frac{IL}{IT}$. The parallel line cuts the triangle's sides proportionally, that's all!



Now let's do a paragraph proof of this…

So, in exercise #3 on p. 298 of Girls Get Curves, we had the following problem:

See the SPLIT diagram. Given: $\overline{PI} \parallel \overline{ST}$. Prove: $\frac{PL}{SL} = \frac{IL}{TL}$ in a two-column proof.



It's almost the same thing as the Side-Splitter Theorem, but if you look carefully at the letters, what we did on p. 298 was to prove that the entire long sides are proportional to the upper segments of the sides. Instead, the Side-Splitter Theorem tells us that the upper segments are proportional to the lower segments: $\frac{PL}{PS} = \frac{IL}{IT}$.

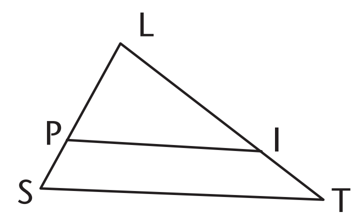
How did we prove $\frac{PL}{SL} = \frac{IL}{TL}$? By using escalators at the mall (transversals) to get corresponding congruent angles, which proved that $\triangle SLT \sim \triangle PLI$, which then proved that we have proportional sides to the triangles: $\frac{PL}{SL} = \frac{IL}{TL}$. (See the solution guide for chapter 17, p.298 for more details!)

As it turns out, what we did on p. 298 is the beginning of the proof of the Side-Splitter Theorem!

See, with a little fancy algebra, we can transform $\frac{PL}{SL} = \frac{IL}{TL}$ (what we already proved) into an equivalent statement, $\frac{PL}{PS} = \frac{IL}{IT}$ (what we want to prove now).

Check it out:

So at this point, we've already proven that $\frac{PL}{SL} = \frac{IL}{TL}$ (so this becomes our "Given"), and we want to prove $\frac{PL}{PS} = \frac{IL}{IT}$.



First, just by looking at the diagram, we can see that $SL = PL + PS$ and that $TL = IL + IT$, right? Just for fun[1], let's substitute these into our Given equation's denominators, and we get:

$$\frac{PL}{SL} = \frac{IL}{TL}$$

$$\rightarrow \quad \frac{PL}{PL+PS} = \frac{IL}{IL+IT}$$

Now this is just a ratio, so we can flip both fractions upside down and we still get a true statement: $\frac{PL+PS}{PL} = \frac{IL+IT}{IL}$.

(This is just like how since $\frac{1}{2} = \frac{3}{6}$ is true, it's also true that $\frac{2}{1} = \frac{6}{3}$).

Looking at just the left side of our equation, $\frac{PL+PS}{PL}$, we can split up the numerator and rewrite this as $\frac{PL}{PL} + \frac{PS}{PL}$.

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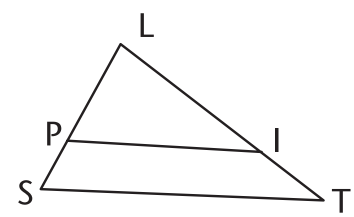
[1] Depends on your definition of "fun."

But lookie there! The two *PL*'s cancel, and this simplifies to $\mathbf{1+\frac{PS}{PL}}$.

(Wait- why could we rewrite $\frac{PL+PS}{PL}$ *as* $\frac{PL}{PL}+\frac{PS}{PL}$*? Well, it's kind of like fraction addition – in reverse! After all, we can add* $\frac{x}{2}+\frac{y}{2}$ *together and get* $\frac{x+y}{2}$*, right? Because after all, they have the same denominator and so we can just add across the top. And we could go in reverse, too: If we were given* $\frac{x+y}{2}$*, we could rewrite it as* $\frac{x}{2}+\frac{y}{2}$ *if we really wanted to.)*

In the same exact way, $\frac{IL+IT}{IL}$ can be rewritten as $\frac{IL}{IL}+\frac{IT}{IL}$ and then as $\mathbf{1+\frac{IT}{IL}}$.

With me so far?



To summarize, our original given, $\frac{PL}{SL}=\frac{IL}{TL}$, has been rewritten like this:

$$\frac{PL}{SL}=\frac{IL}{TL}$$

$\rightarrow$ $\frac{PL+PS}{PL}=\frac{IL+IT}{IL}$ (we did this part on the last page)

$\rightarrow$ $\frac{PL}{PL}+\frac{PS}{PL} = \frac{IL}{IL}+\frac{IT}{IL}$ (on both sides: fraction addition, in reverse!)

$\rightarrow$ $\mathbf{1+\frac{PS}{PL} = 1+\frac{IT}{IL}}$ (simplified two of the fractions)

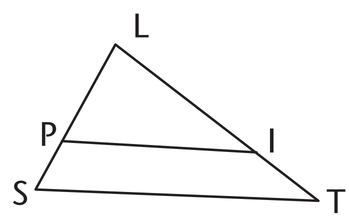
And now we can just subtract 1 from both sides of this equation, and we get $\frac{PS}{PL}=\frac{IT}{IL}$, right? As you know, we can just flip these both upside down and we get an equivalent statement, and that's $\frac{PL}{PS} = \frac{IL}{IT}$. How about that! That's the ratio we were trying to get all along…

$$\therefore\ \boldsymbol{\frac{PL}{PS} = \frac{IL}{IT}}$$

Yes, we've just proven:

The Side-Splitter Theorem: If a line is parallel to one side of a triangle and intersects the other two sides, it divides those two sides proportionally.

So how do we use this theorem? Well, for example, if we were given this SPLIT diagram, told that $\overline{PI} \parallel \overline{ST}$, and then we were given that *PS* = 8, *IT* = 10, and *IL* = 25, we could find *PL*. (Can be used in the architecture of buildings!)



Since $\overline{PI} \parallel \overline{ST}$, we know that $\frac{PL}{PS} = \frac{IL}{IT}$, in other words, we know that $\frac{PL}{8} = \frac{25}{10}$.

Before we take the cross-products, let's reduce the fraction on the right, so we get: $\frac{PL}{8} = \frac{5}{2}$. Ah, much better! And the cross-product gives us: 2*PL* = 40 → ***PL* = 20**.

Much easier to use it than to prove it, right? But these kinds of tricky-algebra proofs are great practice. Every time you read one, you see another trick that could come in handy sometime in the future, making you a force to be reckoned with! ☺