

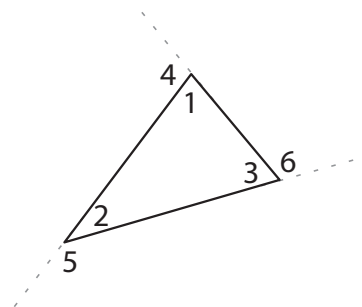
# Girls Get Curves

## Deriving the Sum of Exterior Angles Formula for $n$ -gons

(As promised on p. 213 in *Girls Get Curves*)

Before deriving the sum of exterior angles formula for an  $n$ -gon, let's review how we did it on p. 213 for triangles! It'll make it much easier to follow the  $n$ -gon version, trust me.

Notice that for each vertex of a triangle, the exterior angle and interior angle are supplementary to each other. For example, in this triangle:  $\angle 1 + \angle 4 = 180^\circ$ ,  $\angle 2 + \angle 5 = 180^\circ$ , and  $\angle 3 + \angle 6 = 180^\circ$ . And that means that the sum of all six angles is  $180^\circ + 180^\circ + 180^\circ$ , right?



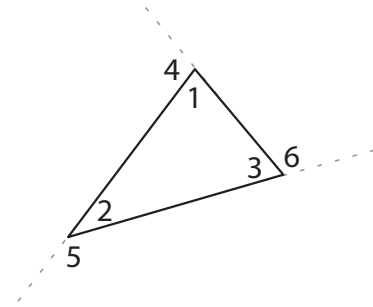
In other words:  $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 = 3(180^\circ)$ . Make sure you follow that before reading on...

We also know that  $\angle 1 + \angle 2 + \angle 3 = 180^\circ$ , because it's a triangle, after all! Now let's substitute  $180^\circ$  for  $\angle 1 + \angle 2 + \angle 3$  in the bolded equation above, and simplify:

$$180^\circ + \angle 4 + \angle 5 + \angle 6 = 3(180^\circ)$$

$$\rightarrow \angle 4 + \angle 5 + \angle 6 = 540^\circ - 180^\circ$$

$$\rightarrow \angle 4 + \angle 5 + \angle 6 = \mathbf{360^\circ}$$



And we've just shown that the sum of all three exterior angles of a *triangle* equals  $360^\circ$ . Nice.



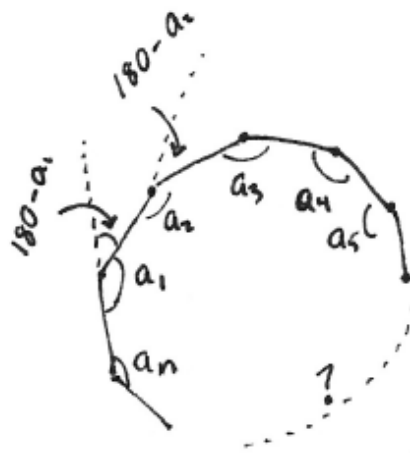
Now we'll do this same thing, but for an  $n$ -gon. Yep, just any old  $n$ -gon. Ready? Let's do it!

We'll start by labeling the interior angles (there are  $n$  of them) like this:

$$a_1, a_2, a_3, \dots, a_n.$$

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<sup>1</sup> We label them with little subscripts because it's a great way to keep track of the fact that there are  $n$  of them – and the “...” is there because we're making a list of these angles, but we don't know how much  $n$  is! These are two tricks you'll see again and again in pre-Calculus.



There are  $n$  angles  
at the  $n$  vertex  
points!  
(But we can't draw  
them all because we  
don't know how big  
"n" is.)

Because it's an  $n$ -gon, we know from p. 208 in *Girls Get Curves* (and that fact that we could draw  $n - 3$  diagonals to create  $n - 2$  triangles) that the sum of these interior angles equals  $180(n - 2)$ , right? That's just **180** degrees for each of the  $(n - 2)$  triangles. (To remember why this works, re-read p. 204-205.)

And that means we can write this:

$$\mathbf{a_1 + a_2 + a_3 + \dots + a_n = 180(n - 2)}$$

Just breathe.

After all, both sides of that big bold equation are just two ways of writing **the sum of all the interior angles of our  $n$ -gon**. It's one thing, written two different ways, so they must be equal to each other. Once you agree, keep reading!

Okay, now do we *know* anything about the exterior angles? Well, sure! For example, we know that the exterior angle at the first vertex is supplementary to  $a_1$ , so that means the first exterior angle's measure is  $180 - a_1$ . And since we can do this for all  $n$  exterior angles, we can write this expression for the sum of the exterior angles of our  $n$ -gon:

$$\text{Sum of Exterior Angles} = (180 - a_1) + (180 - a_2) + (180 - a_3) + \dots + (180 - a_n) = ?$$

Can we somehow show that this always equals 360?

Let's rewrite the above formula by removing the parentheses and combining all of the 180's together. Notice that there must be  $n$  of them; one for each of the  $n$  angles they're paired with. And what do we get if we add " $n$ " 180's together? We get  $180n$ , of course!<sup>2</sup> Now we can rewrite the above expression as:

$$\text{Sum of Exterior Angles} = 180n - a_1 - a_2 - a_3 - \dots - a_n = ?$$

Now, let's factor out a negative sign from each of those  $a$ 's, and we get:

$$\text{Sum of Exterior Angles} = 180n - [a_1 + a_2 + a_3 + \dots + a_n] = ?$$

(Just try distributing the negative sign back into the brackets, and you'll see how that worked.) Any hey, notice that this big thing in the brackets is the *sum of the interior angles*. How about that!

In the bolded equation (on the last page), we saw that this stuff inside the brackets equals  $180(n - 2)$ , so we can make a substitution:

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<sup>2</sup> After all, if we add three 5's together, we get  $3 \times 5$ , right? This is the *same thing*.

**Sum of Exterior Angles =**

$$\begin{aligned} & 180n - [a_1 + a_2 + a_3 + \dots + a_n] \\ &= 180n - [180(n - 2)] \\ &= 180n - [180n - 360] \\ &= 180n - 180n + 360 \\ &= \mathbf{360!} \end{aligned}$$

We know these are equal because of the big bold equation on the previous page!

I think it's kind of amazing that all the angles and – even the  $n$ 's – canceled away, and we're left with the result that *no matter how many angles* are in a polygon, the sum of the exterior angles will always be  $360^\circ$ .



Even if you only followed parts of that derivation, good for you! Try reading it again sometime. This stuff isn't easy, and just by reading it and reasoning it out, you're becoming a much more powerful mathematician.

I'm, um, ridiculously proud of you.